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FREE BOUNDARY REGULARITY FOR SURFACES MINIMIZING Area(S) + c Area(∂S)

BY

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ABSTRACT. In $\mathbf{R}^n$, fix a hyperplane Z and a $(k-1)$-dimensional surface F lying to one side of Z with boundary in Z. We prove the existence of S and B minimizing $\operatorname{Area}(S) + c\operatorname{Area}(B)$ among all k-dimensional S having boundary $F \cup B$, where B is a free boundary constrained to lie in Z. We prove that except possibly on a set of Hausdorff dimension $k-2$, S is locally a $C^{1,\alpha}$ manifold with $C^{1,\alpha}$ boundary B for $0 < \alpha < 1/2$. If $k = n-1$, $C^{1,\alpha}$ is replaced by real analytic.

1. Introduction. This paper establishes the existence and analyticity at the free boundary of surfaces which solve a variational problem where a weighted average of the area of the surface and the area of the boundary is minimized.

Our variational problem is described as follows: We fix a positive real number c and let k and n be integers such that $2 < k < n$. In the ambient space $\mathbf{R}^n = \mathbf{R}^{k-1} \times \mathbf{R} \times \mathbf{R}^{n-k}$ we fix a hyperplane $Z = \mathbf{R}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$ and a $(k-1)$-dimensional surface F of finite area whose nonempty boundary is contained in Z. We consider k-dimensional surfaces S having a boundary which is the union of F and a $(k-1)$-dimensional surface B, where B is a free boundary constrained to lie in Z. For such S and B we measure the quantity

$$\operatorname{Area}(S) + c\operatorname{Area}(B).$$

A solution to our problem would be a pair S and B, for which this quantity assumes its minimum value.

We establish the existence of a solution in a measure theoretic sense. For such a solution, we prove there exists a set whose m-dimensional Hausdorff measure is zero whenever m exceeds $k-2$ such that if p is a point of the free boundary B in the complement of this set, the following two statements hold.

(1) *There exists an open ball about p in which S is a $C^{1,\alpha}$ manifold with $C^{1,\alpha}$ boundary B, whenever $0 < \alpha < 1/2$.*

(2) *If the codimension of S in n-space is one, then near p, S is an analytic manifold with analytic boundary B.*

Using a hodograph transformation similar to that used in **[10]** our proof extends to the case where Z is a real analytic graph over $\mathbf{R}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$.

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This problem bears some similarity to free boundary problems considered by J. C. C. Nitsche, S. Hildebrant [9], R. Courant, H. Lewy, W. Jäger and others. These problems typically start with a fixed curve in $\mathbf{R}^3$ having its endpoints in a two-dimensional surface Z and consider two-dimensional surfaces S which are bounded by F and Z and minimize area subject to that constraint. The solutions minimize area alone. In our problem, a solution S minimizes area on its interior and along the fixed boundary. At the free boundary, however, both the surface S and the free boundary B attempt to minimize their own areas simultaneously. It is this interplay which is for us at the same time both a source of difficulty and an advantage.

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2. The main result. We follow the standardized terminology of geometric measure theory found in [**8** and **4**].

For $m \geqslant 1$, $U^m(p, r)$ and $B^m(p, r)$ will denote, respectively, the open and closed balls with radius r and center p. When referring to a ball in our ambient space $\mathbf{R}^n$, we will often suppress the n and write $U(p, r)$ or $B(p, r)$.

MAIN THEOREM. *Suppose k and n are integers, $2 < k < n$, and c is a positive real number. Suppose $F \in \mathbf{I}^2_{k-1}(\mathbf{R}^n)$, F has finite mass, $\varnothing \neq \operatorname{spt}^2(\partial F) \subset \mathbf{R}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$ and $\operatorname{spt}^2(F) \sim \operatorname{spt}^2(\partial F) \subset \mathbf{R}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$. Then the following conclusions hold*:

Existence. Among all $\mathscr{B} \in \mathbf{I}^2_{k-1}(\mathbf{R}^n)$ with support in $\mathbf{R}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$ and $\partial\mathscr{B} = -\partial F$, and $\mathscr{S} \in \mathbf{I}^2_k(\mathbf{R}^n)$ with $\partial\mathscr{S} = \mathscr{B} + F$, there exists a pair $\mathscr{B}$, $\mathscr{S}$ which minimizes $\mathbf{M}^2(\mathscr{S}) + c\mathbf{M}^2(\mathscr{B})$.

Regularity. Let $B = \operatorname{spt}^2(F + \mathscr{B}) \sim \operatorname{spt}^2(F)$ and $S = \operatorname{spt}^2(\mathscr{S})$.

(1) *Suppose $p \in B$ such that $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, p) = 1$. Then whenever $0 < \alpha < 1/2$ there exists $r > 0$ such that $S \cap U(p, r)$ is a $C^{1,\alpha}$ manifold with $C^{1,\alpha}$ boundary $B \cap U(p, r)$.*

(2) *Suppose $p \in B$ such that $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, p) = 1$ and $\Theta^k(\mathscr{H}^k \llcorner S, p) = 1/2$. In case $n = k + 1$, whenever $0 < \alpha < 1/2$ there exists an $r > 0$ such that $S \cap U(p, r)$ is a $C^{2,\alpha}$ manifold with $C^{2,\alpha}$ boundary $B \cap U(p, r)$.*

(3) *In case $n = k + 1$, if p is as in* (2), *then there exists an $r > 0$ such that $S \cap U(p, r)$ is a real analytic manifold with real analytic boundary $B \cap U(p, r)$.*

(4) *Let J be the set of points in B which do not satisfy the density hypotheses of* (2). *Then $\mathscr{H}^{k-2+\varepsilon}(J) = 0$ for every $\varepsilon > 0$.*

We anticipate the extension of (2) and (3) to higher codimension. The restriction is due to the limitations of the Schauder estimate used in proving $C^{2,\alpha}$ regularity. An extension of this estimate to systems of partial differential equations and boundary conditions like the unpublished extension mentioned in [**10**] should be applicable here.

Using a hodograph transformation our proof extends to the case where the free boundary is constrained to lie in the graph of a real analytic function over $\mathbf{R}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$.

Throughout this paper, we fix k, n and c as in the statement of the Main Theorem. We also will continue to use the notation $\mathscr{B}$, $\mathscr{S}$ for a flat chain modulo two solution and B, S for a set representation of the solution. Considering $\mathbf{R}^n$ as $\mathbf{R}^{k-1} \times \mathbf{R} \times \mathbf{R}^{n-k}$, we fix the hyperplane $Z = \mathbf{R}^{k-1} \times \{0\} \times \mathbf{R}^{n-k}$. The orthogonal projection $q\colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ will be defined by $q(x_1,\ldots,x_n) = (x_1,\ldots,x_{k-1}, x_{k+1},\ldots,x_n)$ and $q^*\colon \mathbf{R}^{n-1} \to \mathbf{R}^n$ will be the corresponding orthogonal injection whose image is Z.

3. Existence. The proof of existence is a standard argument in which the compactness theorem [**8**, 4.2.26, (4.2.17)$^\nu$] is applied to a sequence $(\mathscr{B}_i, \mathscr{S}_i)$ in $\mathbf{I}^2_{k-1}(\mathbf{R}^n) \times \mathbf{I}^2_k(\mathbf{R}^n)$ such that for each i, $\operatorname{spt}^2(\mathscr{B}_i) \subset Z$, $\partial\mathscr{B}_i = -\partial F$, $\partial\mathscr{S}_i = \mathscr{B}_i + F$ and $\mathbf{M}^2(\mathscr{S}_i) + c\mathbf{M}^2(\mathscr{B}_i)$ decreases to the finite number

$$\inf\left\{\mathbf{M}^2(\mathscr{S}) + c\mathbf{M}^2(\mathscr{B})\colon \operatorname{spt}^2(\mathscr{B}) \subset Z, \partial\mathscr{B} = -\partial F, \partial\mathscr{S} = \mathscr{B} + F\right\}.$$

Note that solutions $\mathscr{S}$ and $\mathscr{B}$ satisfy the following minimizing properties.

(1) $\mathbf{M}^2(\mathscr{S}) + c\mathbf{M}^2(\mathscr{B}) \leqslant \mathbf{M}^2(T) + c\mathbf{M}^2(Q)$ whenever $T \in \mathbf{I}^2_k(\mathbf{R}^n)$ and $Q \in \mathbf{I}^2_{k-1}(\mathbf{R}^n)$ such that $\operatorname{spt} Q \subset Z$, $\partial Q = -\partial F$ and $\partial T = Q + F$.

(2) $\mathbf{M}^2(\mathscr{S}) \leqslant \mathbf{M}^2(T)$ whenever $T \in \mathbf{I}^2_k(\mathbf{R}^n)$ and $\partial T = \mathscr{B} + F$.

4. Other representations of the solution. For later convenience in the proof of regularity, we associate with $\mathscr{S}$ and $\mathscr{B}$ currents and varifolds as follows.

Currents. By the definition of $\mathbf{I}^2_k(\mathbf{R}^n)$ and $\mathbf{I}^2_{k-1}(\mathbf{R}^n)$ there exist $S^* \in \mathscr{R}_k(\mathbf{R}^n)$ and $B^* \in \mathscr{R}_{k-1}(\mathbf{R}^n)$ such that $(S^*)^2 = \mathscr{S}$ and $(B^*)^2 = \mathscr{B}$. We may assume that both S^* and B^* are representative modulo two [**8**, 4.2.26, p. 430], i.e. $\|S^*\| \leqslant \mathscr{H}^k$ and $\|B^*\| \leqslant \mathscr{H}^{k-1}$, or equivalently $\Theta^k(\|S^*\|, x) \leqslant 1$ and $\Theta^{k-1}(\|B^*\|, x) \leqslant 1$ for $\mathscr{H}^k$ and $\mathscr{H}^{k-1}$ almost all x, respectively.

By [**8**, 4.1.28 and 4.2.26, pp. 427, 430], these currents satisfy the following properties.

(1) $(\partial S^*)^2 = \partial(S^*)^2 = \mathscr{B} + F$.

(2) $\Theta^k(\|S^*\|, x) = 1$ for $\|S^*\|$ almost all x.

(3) $\|S^*\| = \|S^*\|^2 = \|\mathscr{S}\|^2 = \mathscr{H}^k \llcorner \{x\colon \Theta^k(\|S^*\|, x) = 1\}$.

(4) $\operatorname{spt}\|S^*\| = \operatorname{spt}^2(\mathscr{S})$.

Analogous statements hold for B^*.

The minimizing properties of $\mathscr{S}$ and $\mathscr{B}$ imply the following minimizing properties of S^* and B^*.

(5) If $T \in \mathscr{R}_k(\mathbf{R}^n)$ and $Q \in \mathscr{R}_{k-1}(\mathbf{R}^n)$ such that $((T)^2 - \mathscr{S})\llcorner(\mathbf{R}^n \sim U) = 0$ and $((Q)^2 - \mathscr{B})\llcorner(\mathbf{R}^n \sim U) = 0$ for some open set U disjoint from $\operatorname{spt}^2(F)$, then $\mathbf{M}(S^*) + c\mathbf{M}(B^*) \leqslant \mathbf{M}^2(T) + c\mathbf{M}^2(Q)$.

(6) S^* is area minimizing modulo two.

(7) $\mathbf{M}(S^*) \leqslant \mathbf{M}(T)$ whenever $T \in \mathscr{R}_k(\mathbf{R}^n)$ with $\partial T = \partial S^*$.

This last property yields

(8) $\|S^*\| = \mathscr{H}^k \llcorner \operatorname{spt}\|S^*\| = \mathscr{H}^k \llcorner S$, and

(9) $\|B^*\| = \mathscr{H}^{k-1} \llcorner \operatorname{spt}\|B^*\| = \mathscr{H}^{k-1} \llcorner B$.

PROOF OF (8). By [8, 5.1.6], there exists $t > 0$ such that $\Theta^k(\|S^*\|, a) \geqslant t$ and $\mathrm{Tan}^k(\|S^*\|, a) = \mathrm{Tan}(\mathrm{spt}\, S^*, a)$ whenever $a \in \mathrm{spt}\|S^*\| \sim \mathrm{spt}\, \partial S^*$. We now argue that $\mathrm{spt}\|S^*\|$ is $(\mathscr{H}^k, k)$-rectifiable. By [8, 3.3.13], $\mathrm{spt}\|S^*\| = W \cup Q$, where Q is $(\mathscr{H}^k, k)$-rectifiable and $\mathscr{H}^k$ measurable, $Q \cap W$ is empty and W is purely $(\mathscr{H}^k, k)$-unrectifiable. Suppose $x \in \mathrm{spt}\|S^*\|$ such that $\Theta^k(\mathscr{H}^k \llcorner Q, x) = 0$, $\Theta^k(\mathscr{H}^k \llcorner W, x) = b > 0$ and such that $\mathrm{Tan}^k(\|S^*\|, x)$ is in $\mathbf{G}(n, k)$. Since the projection of W onto almost all members of $\mathbf{G}(n, k)$ has measure zero, one may use the projection of $\mathrm{spt}\|S^*\|$ onto $\mathrm{Tan}^k(\|S^*\|, x)$ or onto a rotation of this k-plane through a small angle, to construct a comparison current of less mass than S^*, a contradiction.

Now suppose $D \subset \mathrm{spt}\|S^*\|$. Let $\{V_i\}$ be a decreasing sequence of open sets each of which contain D with limit D. By [8, 2.10.19(3)],

$$\|S^*\|(D) = \lim_{i \to \infty} \|S^*\|(V_i) \geqslant t\mathscr{S}^k(D),$$

where $\mathscr{S}^k$ is k-dimensional spherical measure. Therefore, by [8, 2.10.2], $t\mathscr{H}^k(D) \leqslant t\mathscr{S}^k(D) \leqslant \|S^*\|(D)$. Combining this with the previous properties, we obtain (8). (9) is proved by a similar argument.

Varifolds. We define varifolds $v(S) \in V_k(\mathbf{R}^n)$ and $v(B) \in V_{k-1}(\mathbf{R}^n)$ by

$$v(S)(A) = \mathscr{H}^k\big(S \cap \{x\colon (x, \mathrm{Tan}^k(\mathscr{H}^k \llcorner S, x)) \in A\}\big)$$

whenever $A \subset \mathbf{R}^n \times \mathbf{G}(n, k)$ and

$$v(B)(A) = \mathscr{H}^{k-1}\big(B \cap \{x\colon (x, \mathrm{Tan}^{k-1}(\mathscr{H}^{k-1} \llcorner B, x)) \in A\}\big)$$

whenever $A \subset \mathbf{R}^n \times \mathbf{G}(n, k-1)$.

It is easily verified that $\|v(S)\| = \|\mathscr{S}\|^2$ and $\|v(B)\| = \|\mathscr{B}\|^2$.

5. Regularity.

5.1. *Outline of the proof.* We focus our attention on a point p in $B \cap S$ with $\rho = \mathrm{dist}(p, \mathrm{spt}^2(F)) > 0$. By translation we may assume that p is the origin. The proof of regularity is in three parts. (1) First, we assume $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1$ and use the $(\mathbf{M}, \varepsilon, \delta)$-minimality of B and a regularity theorem of F. J. Almgren to show that locally B is the graph of a $C^{1,\alpha}$ function provided $0 < \alpha < 1/2$. (2) We then assume in addition that $\Theta^k(\mathscr{H}^k \llcorner S, 0) = \frac{1}{2}$. After developing the structure of S as a varifold, we use this structure to prove S is a $C^{1,\alpha}$ manifold with $C^{1,\alpha}$ boundary in a neighborhood of the origin whenever $0 < \alpha < 1/2$. (3) Restricting to the case when S has codimension one, we express S locally as the graph of a function over a k-dimensional half-plane. The minimizing properties of the surface and boundary imply that this function satisfies an elliptic partial differential equation and complementing boundary condition. A standard difference quotient argument and Schauder estimate implies the local $C^{2,\alpha}$ regularity of S and B. A regularity theorem of Morrey then allows us to conclude that the surface and boundary are analytic near the origin.

In §6, we return to arbitrary codimension and estimate the size of the set of points in B where our density hypotheses fail.

5.2. *$C^{1,\alpha}$ regularity of B.* In this section we consider B as a $(k-1)$-dimensional subset of $\mathbf{R}^{n-1}$.

THEOREM. *Suppose* $0 < \alpha < 1/2$, $B \subset \mathbf{R}^{n-1}$ *is the free boundary of a solution to our variational problem*, $0 \in B \sim \operatorname{spt}^2(F)$ *and* $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1$. *Then there exists an* R, $0 < R < \operatorname{dist}(0, \operatorname{spt}^2(F))$, *and a rotation* θ *of* $\mathbf{R}^{n-1}$ *such that* $\theta(B \cap U^{n-1}(0, R))$ *is the graph of a* $C^{1,\alpha}$ *function* $f\colon U \to \mathbf{R}^{n-k}$, *where* U *is an open connected subset of* $U^{k-1}(0, R)$ *containing the origin of* $\mathbf{R}^{k-1}$.

PROOF. Let δ be a small positive number such that

$$0 < \delta < \min\left\{ ck/4, \rho = \operatorname{dist}(0, \operatorname{spt}^2(F))\right\}.$$

Define $\xi\colon \mathbf{R}^+ \to \mathbf{R}^+$ by $\xi(t) = (8/ck)t$. This function is nondecreasing and approaches zero as t approaches zero from the positive side.

We shall show that $B \cap U^{n-1}(0, r)$ is $(\mathbf{M}, \xi, \delta)$-minimal with respect to $\mathbf{R}^{n-1} \sim U^{n-1}(0, r)$ for any r satisfying $0 < r < \operatorname{dist}(0, \operatorname{spt}^2(F)) = \rho$ (see Almgren [**6**, III.1 and II.1]), and then apply Almgren's regularity theorem for $(\mathbf{M}, \xi, \delta)$-minimal sets (see [**6**, IV.13(5)]).

Fix such an r. The only condition of $(\mathbf{M}, \xi, \delta)$-minimality which is nontrivial to verify is that

$$\mathscr{H}^{k-1}(B \cap W) \leqslant (1 + \xi(t))\mathscr{H}^{k-1}(\phi(B \cap W))$$

whenever $\phi\colon \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ is a Lipschitz map with $W = \mathbf{R}^{n-1} \cap \{z\colon \phi(z) \neq z\}$, $t = \operatorname{diam}(W \cap \phi(W)) < \delta$, and $\operatorname{dist}(W \cup \phi(W), \mathbf{R}^{n-1} \sim U^{n-1}(0, r)) > 0$. This follows from the minimizing property of B and S. Indeed, fix $b \in W \cup \phi(W)$. Then $W \cup \phi(W) \subset B^{n-1}(b, 2t)$. For some $\varepsilon > 1$, define $h\colon (-\varepsilon, \varepsilon) \times \mathbf{R}^{n-1} \to \mathbf{R}^{n-1}$ by $h(s, x) = (1 - s)b + sx$. By the minimizing property of S^* and B^*,

$$\mathbf{M}(S^*) + c\mathbf{M}(B^*) \leqslant \mathbf{M}\big(S^* + h_\#([0,1] \times (\phi_\#(B^*) - B^*))\big) + c\mathbf{M}(\phi_\#(B^*)).$$

Applying formulas of Federer [**8**, 4.1.9 and 1.7.5], we obtain the following inequalities:

$$\begin{aligned}
c\mathbf{M}(B^* \llcorner W) + c\mathbf{M}(B^* \llcorner (\mathbf{R}^{n-1} \sim W)) &= c\mathbf{M}(B^*) \\
&\leqslant \mathbf{M}(h_\#([0,1] \times (\phi_\#(B^*) - B^*))) + c\mathbf{M}(\phi_\#(B^*)) \\
&\leqslant \int \int_0^1 |x - b| s^{k-1}\, ds\, d\|\phi_\#(B^*) - B^*\| x + c\mathbf{M}(\phi_\#(B^*)) \\
&\leqslant (2t/k)\mathbf{M}(\phi_\#(B^*) - B^*) + c\mathbf{M}(\phi_\#(B^*)) \\
&\leqslant (2t/k)\mathbf{M}(\phi_\#(B^* \llcorner W)) + (wt/k)\mathbf{M}(B^* \llcorner W) \\
&\quad + c\mathbf{M}\big(B^* \llcorner (\mathbf{R}^{n-1} \sim W)\big) + c\mathbf{M}(\phi_\#(B^* \llcorner W)).
\end{aligned}$$

Rearranging terms and noting that $t < ck/4$, we obtain

$$\begin{aligned}
\mathbf{M}(B^* \llcorner W) &\leqslant (1 + 4t/(ck - 2t))\mathbf{M}(\phi_\#(B^* \llcorner W)) \\
&\leqslant (1 + \xi(t))\mathscr{H}^{k-1}(\phi_\#(B^* \llcorner W)).
\end{aligned}$$

The relationship of B and B^* yields the desired inequality.

We are now in a position to apply Almgren's regularity theorem. By hypothesis, $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1$. It is a consequence of $(\mathbf{M}, \xi, \delta)$-minimality (see Taylor [**14**, II.2]) that $\operatorname{Tan}^{k-1}(\mathscr{H}^{k-1} \llcorner B \cap U(0, \rho/2), 0)$ is an area minimizing cone. The density hypothesis, together with Federer [8, 5.4.7], implies this cone is an element of $\mathbf{G}(n-1, k-1)$. For an appropriate rotation θ of $\mathbf{R}^{n-1}$, $B_0 = \theta(B \cap U^{n-1}(0, \rho/2))$ has $\mathbf{R}^{k-1} \times \{0\}$ as its tangent space. B_0 is $(\mathbf{M}, \xi, \delta)$-minimal with respect to $\mathbf{R}^{n-1} \sim U^{n-1}(0, \rho/2)$. Direct computation yields

$$\int_0^1 t^{-(1+\alpha)} \xi(t)^{1/2}\, dt < \infty$$

when $0 < \alpha < 1/2$. Let s be the constant s of [6, IV.8]. It is easily verified that there exists a small positive r as required in [6, IV.13(5)] so that Almgren's theorem implies the existence of a $C^{1,\alpha}$ function

$$f\colon B^{k-1}(0, sr/2) \to \mathbf{R}^{n-k}$$

such that

$$\operatorname{graph}(f) = \theta\big(B \cap U^{n-1}(0, \rho/2)\big) \cap \big[B^{k-1}(0, sr/2) \times \mathbf{R}^{n-k}\big].$$

Let $\mathscr{R} = \min\{\rho/2, sr/2\}$.

5.3. *Varifold structure of* S. We now prove some variational inequalities, monotonicity results and density results for the varifold $v(S)$. These results will be needed to prove $C^{1,\alpha}$ regularity of the surface S at the free boundary for $0 < \alpha < 1/2$.

IMPORTANT REMARK. While the propositions in this section are stated for $v(S)$ the reader will note that the results depend only on the fact that $v(S)$ is associated with a rectifiable current S^* for which the following properties hold:

(1) S^* is representative modulo two.

(2) For $0 < R < \rho = \operatorname{dist}(0, \operatorname{spt}^2(F))$,

$$\mathbf{M}^2(S^* \llcorner U(0, R)) + c\mathbf{M}^2(\partial S^* \llcorner U(0, R)) \leqslant \mathbf{M}^2(T \llcorner U(0, R)) + c\mathbf{M}^2(\partial T \llcorner U(0, R))$$

whenever S^* and T agree on the complement of $U(0, R)$.

(2′) In particular,

$$\mathbf{M}^2(S^* \llcorner U(0, R)) \leqslant \mathbf{M}^2(T \llcorner U(0, R))$$

whenever S^* and T agree on the complement of $U(0, R)$ and $\partial S^* = \partial T$.

(3) There exists an $R > 0$ such that $(\operatorname{spt} \partial S^*) \cap U(0, R)$ is a $(k-1)$-dimensional $C^{1,\alpha}$ manifold of $\mathbf{R}^n$.

The proof of Theorem 5.4 will draw on this remark.

5.3.1. *Variational properties*. Let U be an open subset of $\mathbf{R}^n$ which is disjoint from $\operatorname{spt}^2(F)$. Suppose $g\colon \mathbf{R}^n \to \mathbf{R}^n$ is a smooth vector field with support in U such that $g(Z) \subset Z$. For some $\varepsilon > 0$, let $h\colon (-\varepsilon, \varepsilon) \times \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map such that $h(0, x) = x$, $g(x) = (\partial/\partial t)|_{t=0}(h(t, x))$ for all $x \in \mathbf{R}^n$, and $\{x\colon h(t, x) \neq x$ for some t in $(-\varepsilon, \varepsilon)\}$ has compact closure in $\mathbf{R}^n$. For each t in $(-\varepsilon, \varepsilon)$, let $h_t\colon \mathbf{R}^n \to \mathbf{R}^n$ be defined by $h_t(x) = h(t, x)$. This deformation may be used in two ways to construct comparison surfaces to S^*, B^*.

First, S^*, B^* may be compared to $h_{t\#}(S^*)$, $h_{\#}(B^*)$. Define the function $f\colon (-\varepsilon, \varepsilon) \to \mathbf{R}$ by

$$F(t) = \|h_{t\#} v(S)\|(U) + c\mathbf{M}^2(h_{t\#} B^*).$$

By the minimality properties of S^* and B^*, F assumes its minimum when t is zero. Differentiating F yields the following result.

PROPOSITION 1. *If* $g\colon \mathbf{R}^n \to \mathbf{R}^n$ *is a smooth vector field with support disjoint from* $\operatorname{spt}^2(F)$ *such that* $g(Z) \subset Z$, *then*

$$0 = \delta(v(S))(g) + c(d/dt)|_{t=0}\mathbf{M}^2(h_{t\#}B^*).$$

This has an immediate corollary.

COROLLARY 2. *Let* U *be an open subset of* $\mathbf{R}^n$ *which is disjoint from* $\operatorname{spt}^2(F)$. *Then* $v(S)$ *is stationary in* $U \sim B$.

Second, with g and h as before and $\varepsilon > 1$, S^*, B^* may also be compared with the pair $h_{t\#}(S^*) + h_{\#}([0, t] \times B^*)$, B^*. Define $f\colon (0, \varepsilon) \to \mathbf{R}$ by

$$f(t) = \|h_{t\#}v(S)\|(\mathbf{R}^n) + \mathbf{M}(h_{\#}([0, t] \times B^*)).$$

The minimality of S^*, B^* implies that for t in $(0, \varepsilon)$,

$$\begin{aligned} f(t) &\geqslant \mathbf{M}^2(h_{t\#}S^*) + \mathbf{M}^2(h_{\#}([0, t] \times B^*)) \\ &\geqslant \mathbf{M}^2(h_{t\#}S^* + h_{t\#}([0, t] \times B^*)) \\ &\geqslant \mathbf{M}^2(S^*) = f(0). \end{aligned}$$

Thus the derivative on the right of f at zero is positive and

$$0 \leqslant \delta(v(S))(g) + (d/dt)|_{t=0}\mathbf{M}(h_{\#}([0, t] \times B^*)).$$

Furthermore, using [8, 4.1.9, 4.1.28(5), 1.7.5 and 1.7.6] and the definition of h, we compute that

$$\begin{aligned} &(d/dt)|_{t=0}\mathbf{M}(h_{\#}([0, t] \times B^*)) \\ &\qquad = \int \left|(d/dt)|_{t=0}h_t(x) \wedge \langle T_x, \Lambda_{k-1}Dh_0(x)\rangle\right| d\|B^*\|x \\ &\qquad \leqslant \int |g(x) \wedge \langle T_x, \Lambda_{k-1}\mathbf{1}_{\mathbf{R}^n}\rangle| d\|B^*\|x \\ &\qquad \leqslant \int |g(x)| d\|B^*\|x. \end{aligned}$$

Therefore,

$$0 \leqslant \delta(v(S))(g) + \int |g(x)| d\|B^*\|x.$$

Combining this with a similar computation for $-g$ yields two results.

PROPOSITION 3. *If* g *is a smooth vector field on* $\mathbf{R}^n$ *with support in* $\mathbf{R}^n \sim \operatorname{spt}^2(F)$, *then*

$$|\delta(v(S))(g)| \leqslant \int |g(x)| d\|B^*\|x.$$

COROLLARY 4. $\|\delta(v(S))\|$ *is a Radon measure on* $\mathbf{R}^n \sim \operatorname{spt}^2(F)$.

5.3.2. *Monotonicity*. Next we prove an *isoperimetric* type *inequality* and a *monotonicity* result for $v(S)$ at points *on the free boundary.*

PROPOSITION 1. *If* $0 \in B$, $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1$, $R > 0$ *and* $B \cap U(0, R)$ *is the graph of a* $C^{1,\alpha}$ *function* f: $\operatorname{Tan}(B, 0) \to \operatorname{Tan}(B, 0)^{\perp}$, *then there is an s such that whenever* $b \in U(0, R - S) \cap B$,

$$\|v(S)\| U(b, r) \geqslant [\gamma(d+1)k]^{-k} r^k,$$

where γ *is the isoperimetric constant of* [**8**, 4.2.10] *and d is a constant depending on* n, k *and the Hölder constant of* Df.

PROOF. It is convenient to prove this proposition for $\|S^*\|$ which is equal to $\|v(S)\|$.

As $B \cap U(0, R)$ is the graph of a $C^{1,\alpha}$ function, there exists $0 < s < R$ and E such that $\|B^*\| U(b, r) \leqslant E r^{k-1} \alpha(k-1)$ for $0 < r < s$ and b in $U(0, R - s)$. Fix b.

Define u: $\mathbf{R}^n \to \mathbf{R}$ by $u(x) = |x - b|$ and let $\langle S^*, u, r\rangle$ denote the slice of S^* on the $\partial B(b, r)$ (see [**8**, 4.2.1]). Also define m: $(0, s) \to \mathbf{R}$ by $m(r) = \|S^*\| U(b, r)$.

By [**8**, 4.2.1],

$$\mathbf{M}(\langle S^* u, r\rangle) \leqslant m'(r). \tag{1}$$

Next we prove there is a d such that

$$\|B^*\| U(b, r) \leqslant d\, \mathbf{M}(\langle S^*, u, r\rangle) \tag{2}$$

whenever $0 < r < s$. Let Y denote the tangent space to B at the origin. Y is the image of $\mathbf{R}^{k-1} \times \{0\}$ under a rotation θ of $\mathbf{R}^n$. Let $\tau(b)$: $\mathbf{R}^n \to \mathbf{R}^n$ be the translation which takes b to the origin and p_{k-1}: $\mathbf{R}^n \to \mathbf{R}^{k-1}$ be defined by by $p_{k-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_{k-1})$. Now

$$\partial[\tau(b)_{\#} \langle S^*, u, r\rangle] = \partial[\tau(b)_{\#}(B^* \llcorner U(b, r))]$$

has support contained in $\{z: |Y(z)| > r - \beta\}$ for some $\beta \leqslant Cr$, where C depends on the Hölder constant of the derivative of the function f: $Y \to Y$ of which B is the graph. Thus $(p_{k-1} \circ \theta^{-1} \circ Y \circ \tau(b))_{\#} \langle S^*, u, r\rangle$ is a top dimensional flat chain whose boundary has support outside $U^{k-1}(0, r - \beta)$. By the constancy theorem [**8**, 4.1.7] there exists a number a such that

$$\operatorname{spt}\big((p_{k-1} \circ \theta^{-1} \circ Y \circ \tau(b))_{\#} \langle S^*, u, r\rangle - a\mathbf{E}^{k-1} \llcorner U(0, r - \beta)\big)$$

is contained in $\mathbf{R}^{k-1} \sim U^{k-1}(0, r - \beta)$, where $\mathbf{E}^{k-1}$ is the flat chain associated with $\mathbf{R}^{k-1}$. Combining this with our original assumption on β, we obtain

$$\begin{aligned} \mathbf{M}(\langle S^*, u, r\rangle) &\geqslant \operatorname{Lip}(p_{k-1} \circ \theta^{-1} \circ Y \circ \tau(b)) \mathbf{M}(a\mathbf{E}^{k-1} \llcorner U^{k-1}(0, r - \beta)) \\ &\geqslant a(r - \beta)^{k-1} r^{-(k+1)} E^{-1} 2^{-1} \|B^*\| U(b, r) \\ &\geqslant 2^{-1}(1 - C)^{k-1} a E^{-1} \|B^*\| U(b, r). \end{aligned}$$

Let $d = 2a^{-1}(1 - C)^{1-k} E$.

Now we compute a lower bound for $r^{-k}m(r)$. By the isoperimetric inequality [**8**, 4.2.10], there exists a constant γ depending only on k and n such that

$$\begin{aligned}\mathbf{M}(S^*\llcorner U(b,r))^{(k-1)/k} &\leqslant \gamma\mathbf{M}(\partial[S^*\llcorner U(b,r)])\\ &\leqslant \gamma\mathbf{M}(\langle S^*,u,r\rangle) + \gamma\mathbf{M}(B^*\llcorner U(b,r)).\end{aligned}$$

Applying (1) and (2) and rearranging terms, one obtains

$$[\gamma(d+1)]^{-1} \leqslant [m(r)]^{(-k+1)/k} m'(r).$$

The proof is completed by integrating with respect to r and raising the result to the kth power.

PROPOSITION 2. *If* $0 \in B$, $\Theta^{k-1}(\mathscr{H}^{k-1}\llcorner B, 0) = 1$ *and* $R > 0$ *such that* $B \cap U(0, R)$ *is the graph of a* $C^{1,\alpha}$ *function* f: $\operatorname{Tan}(B,0) \to \operatorname{Tan}(B,0)^{\perp}$, *then there is an* s *and a constant* D, *depending on* k, n, *and the Hölder constant of* Df *such that*

$$r^{-k}\|v(S)\|U(b,r)\exp(\alpha^{-1}Dr^{\alpha})$$

is a nondecreasing function of r *for* $0 < r < s$ *whenever* $b \in U(0, R-s) \cap B$.

PROOF. Let S, E, and m be as in the proof of Proposition 1. For a fixed $\varepsilon > 1$, define h: $(-\varepsilon, \varepsilon) \times \mathbf{R}^n \to \mathbf{R}^n$ by $h(t,x) = tx$. Then whenever $0 < r < s$,

$$S^*\llcorner(\mathbf{R}^n \sim U(b,r)) + h_{\#}([0,1] \times \langle S^*, u, r\rangle) + h_{\#}([0,1] \times (B^*\llcorner B(b,r)))$$

has boundary ∂S^* and agrees with S^* outside $U(b,r)$. Since S^* is area minimizing,

$$m(r) \leqslant \mathbf{M}(h_{\#}([0,1] \times \langle S^*, u, r\rangle)) + \mathbf{M}(h_{\#}([0,1] \times (B^*\llcorner B(b,r)))).$$

Since $\operatorname{spt}\langle S^*, u, r\rangle \subset \partial B(b,r)$, the first term on the right does not exceed $(r/k)\mathbf{M}(\langle S^*, u, r\rangle)$. We shall find a constant A such that the second term does not exceed $(1/k)Ar^{1+\alpha}\|B^*\|U(b,r)$. By (1) of the proof of Proposition 1, we would then have

$$m(r) \leqslant (r/k)m'(r) + (A/k)r^{1+\alpha}\|B^*\|U(b,r).$$

Multiplying by $k[m(r)r]^{-1}$, applying Proposition 1 and recalling the original assumption on r and s then yields

$$k/r \leqslant m'(r)/m(r) + Dr^{\alpha-1},$$

where D is a constant depending on k, n and the Hölder constant of Df. Integrating and rearranging terms we obtain

$$0 \leqslant (d/dr)\left(\ell n\left(r^{-k}m(r)\right) + \alpha^{-1}Dr^{\alpha}\right).$$

From this the proposition easily follows.

We now return to find A. Recalling [**8**, 1.4.9 and 4.1.28(5)], we let T_x denote a simple unit $(k-1)$-vector associated to the tangent space of B at x, and compute

$$\begin{aligned}&\mathbf{M}(h_{\#}([0,1] \times B^*\llcorner B(b,r)))\\ &\qquad\leqslant \int_{U(b,r)}\int_0^1 |(d/dt)h_t(x) \wedge \langle T_x, \Lambda_{k-1}Dh_t(x)\rangle|\, d\mathscr{L}^1 t\, d\|B^*\|x\\ &\qquad\leqslant \int_{U(b,r)}\int_0^1 |x \wedge T_x| t^{k-1}\, d\mathscr{L}^1 t\, d\|B^*\|x\\ &\qquad\leqslant (1/k)\int_{U(b,r)} |x \wedge T_x|\, d\|B^*\|x.\end{aligned}$$

We will be done if we can find a constant A such that $|x \wedge T_x| \leqslant A|x|^{1+\alpha}$.

Let $p_{k-1}\colon \mathbf{R}^n \to \mathbf{R}^{k-1}$ and $p_{n-k}\colon \mathbf{R}^n \to \mathbf{R}^{n-k}$ denote projections onto the first $k-1$ coordinates and the last $n-k$ coordinates, respectively. Let $p_{k-1}^*\colon \mathbf{R}^{k-1} \to \mathbf{R}^n$ and $p_{n-k}^*\colon \mathbf{R}^{n-k} \to \mathbf{R}^n$ be the associated orthogonal injections onto $\mathbf{R}^{k-1} \times \{0\}$ and $\{0\} \times \mathbf{R}^{n-k}$.

Each x in $\operatorname{spt}\|B^*\| \cap U(b, r)$ is of the form $(y, f(y))$ for some y. So

$$\begin{aligned}|x \wedge T_x| &\leqslant |x \wedge Tx| \cdot \|\Lambda_{k-1}(p_{k-1}^* \circ \mathbf{1}_{k-1} + p_{n-k}^* \circ Df(y))\| \\ &= |(y, f(y)) \wedge \langle e_1, p_{k-1}^* \circ \mathbf{1}_{k-1} + p_{n-k}^* \circ Df\rangle \\ &\qquad \wedge \cdots \wedge \langle e_{k-1}, p_{k-1}^* \circ \mathbf{1}_{k-1} + p_{n-k}^* \circ Df(y)\rangle|\end{aligned}$$

By definition [**8**, 1.7.5], this norm is equal to the square root of the sum of the squares of the $k \times k$ subdeterminants of

$$\begin{pmatrix} Y & f(y) \\ \mathbf{1}_{k-1} & (Df(y))^T \end{pmatrix}.$$

Thus the form does not exceed the sum of the absolute values of $k \times k$ subdeterminants. Each subdeterminant is in turn a sum of terms of the form $f_i(y)$ or either y_i or $f_i(y)$ times a product of $k-1$ derivatives $D_l f_j(y)$, where $f = (f_1, \ldots, f_{n-k})$ and $y = (y_1, \ldots, y_{k-1})$. We bound each of these terms. Let C be the Hölder constant of Df. Clearly $|y_i| \leqslant |y| \leqslant |x|$, and

$$|D_l f_j(y)| \leqslant |Df(y)| \leqslant n^{1/2}\|Df(y)\| \leqslant n^{1/2} C |x|^\alpha.$$

To bound $|f_i(y)|$ we apply the mean-value theorem to find z on the line between the origin and y such that $f_i(y) = \operatorname{grad} g_i(z) \cdot y$. Thus

$$|f_i(y)| \leqslant |\operatorname{grad} f_i(z)|\,|y| \leqslant n^{1/2} C |z|^\alpha |y| \leqslant n^{1/2} C |x|^{1+\alpha}.$$

Combining these inequalities and recalling that $|x| < 1$, we find that the absolute value of no subdeterminant exceeds $\sup\{n^{1/2}C, n^{(k-2)/2}C^{k-1}, n^{k-2}C^k\}|x|^{1+\alpha}$. It is now easy to obtain a constant A depending on n, k and C such that

$$|x \wedge T_x| \cdot \|\Lambda_k(p_{k-1}^* \circ \mathbf{1}_{k-1} + p_{n-k}^* \circ Df(y))\| \leqslant A|x|^{1+\alpha}.$$

5.3.3. *Density of* $v(S)$. If U is an open subset of $\mathbf{R}^n$ disjoint from $\operatorname{spt}^2(F) \cup B$, then the interior monotonicity implies $\Theta^k(\|v(S)\|, x)$ is real for each x in U. Futhermore, since $\Theta^k(\|v(S)\|, x) = 1$ for almost all x in U (see §4(2)) and $v(S)$ is stationary in U (see Corollary 2 in §5.3.1), the density $\Theta^k(\|v(S)\|, \cdot)$ is *upper semicontinuous* on U by [**4**, 8.6]. In particular, $\Theta^k(\|v(S)\|, x) \geqslant 1$ for all x in $U \cap \operatorname{spt}\|v(S)\|$.

A similar *upper semicontinuity* result *along the free boundary* follows from our boundary monotonicity result by an argument similar to that of [**4**, 5.4].

PROPOSITION 1. *Suppose* $0 < \alpha < 1/2$ *and* U *is an open subset of* $\mathbf{R}^n \sim \operatorname{spt}^2(F)$ *such that* $U \cap B$ *is a* $C^{1,\alpha}$ *manifold. Then* $\Theta^k(\|v(S)\|, \cdot)$ *is a real-valued upper semicontinuous function on* $U \cap B$.

5.3.4. *Varifold tangents to* $v(S)$.

PROPOSITION 1. *Suppose* $0 \in B \cap \operatorname{spt}\|v(S)\| \sim \operatorname{spt}^2(F)$ *such that*

$$\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1 \quad \textit{and} \quad \Theta^k(\mathscr{H}^k \llcorner S, 0) = 1/2.$$

Then

(a) $v(S)$ *has a varifold tangent* C *at* 0;

(b) $\operatorname{Tan}(B,0) \in \mathbf{G}(n, k-1)$; *and*

(c) *there is a unit vector* u *normal to* $\operatorname{Tan}(B,0)$ *such that* C *is the varifold associated to* $\{y + tu \colon y \in \operatorname{Tan}(B,0)$ *and* $t \geqslant 0\}$.

PROOF. Part (a) follows from [**4**, 3.4] since $\Theta^k(\|v(S)\|, 0)$ is finite. Part (b) is a consequence of 5.4. It remains to prove part (c). Let r_i be a sequence of positive numbers which increases to infinity and let

$$C = \lim_{i \to \infty} \mu(r_i)_\# \|v(S)\|,$$

where $\mu(r_i)\colon \mathbf{R}^n \to \mathbf{R}^n$ is multiplication by r_i. Let α be between 0 and 1/2 and let U be an open subset of $\mathbf{R}^n \sim \operatorname{spt}^2(F)$ containing the origin such that $U \cap B$ is a $C^{1,\alpha}$ manifold. Since $\|v(S)\|$ is stationary in $U \sim B$, we apply [**4**, 4.11] to conclude that $\|\delta(v(S))\| \llcorner (\mathbf{R}^n \sim \operatorname{Tan}(B,0)) = 0$. Letting V be C plus its reflection across $\operatorname{Tan}(B,0)$, the Reflection Principle [**5**, 3.2] implies V is stationary. By the interior semicontinuity of the density of $\|v(S)\|$ and [**4**, 5.3], there is a T in $\mathbf{G}(n,k)$ such that $V = 2\Theta^k(\|C\|, 0) v(T)$, where $v(T)$ is the varifold associated with T. Hence $C = 2\Theta^k(\|C\|, 0) v(W)$, where W is a connected component of $T \sim \operatorname{Tan}(B,0)$. Finally, $\Theta^k(\|C\|, 0) = 1/2$ by the hypotheses on the density of $\|v(S)\|$. Part (c) now follows immediately.

REMARK. By translation, this proposition holds for any $b \in B \cap \operatorname{spt}\|v(S)\| \sim \operatorname{spt}^2(F)$ satisfying the density hypotheses.

5.3.5. *Other results*. Let α be between 0 and 1/2. Let U be an open subset of $\mathbf{R}^n \sim \operatorname{spt}^2(F)$ such that $U \cap B$ is a $C^{1,\alpha}$ manifold.

Applying [**4**, 5.4 and 8.6] to $C \in \operatorname{Var}\operatorname{Tan}(v(S), b)$ plus its reflection across $\operatorname{Tan}(B, b)$ one obtains the following.

PROPOSITION 1. $\Theta^k(\|v(S)\|, b) \geqslant 1/2$ *whenever* $b \in U \cap B \cap \operatorname{spt}\|v(S)\|$ *and* $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, b) = 1$.

The argument of [**5**, 3.5(3)] directly implies

PROPOSITION 2. *There is a number* $\mu > 1$ *with the property that if* $b \in U \cap B \cap \operatorname{spt}\|v(S)\|$, $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, b) = 1$ *and* $2\Theta^k(\|v(S)\|, b) < \mu$, *then there exists a positive real number* r *such that* $B \cap U^n(b, r) \subset \operatorname{spt}\|v(S)\|$.

5.4. $C^{1,\alpha}$ *regularity of* S. We now prove that the surface S is a $C^{1,\alpha}$ manifold with boundary $(0 < \alpha < 1/2)$ in a neighborhood of any point on the free boundary B at which the boundary has density one and the surface has density one-half. If we assume that the origin is the point under consideration and for each positive r let $\mu(r^{-1})\colon \mathbf{R}^n \to \mathbf{R}^n$ be the homothety $\mu(r^{-1})(x) = r^{-1}x$, then the density hypotheses imply that for small r, $\|\mu(r^{-1})_\# v(S)\|(U(0,1))$ is close to the mass of a k-dimensional half-disc, and $\operatorname{spt}\|\mu(r^{-1})_\# v(B)\| \cap U(0,1)$ is, by §5.2, the graph over $\operatorname{Tan}(B,0)$

of a $C^{1,\alpha}$ function $(0 < \alpha < 1/2)$ and has area close to that of a $(k-1)$-dimensional disc. The $\mu(r^{-1})_{\#}v(S)$ also maintains the minimizing properties of $v(S)$. We prove that if V is a varifold satisfying the above properties, then $\operatorname{spt}\|V\| \cap U(0,1)$ is the graph over a k-dimensional half-plane of a $C^{1,\alpha}$ function.

Fix α, $0 < \alpha < 1$. Whenever N is a $(k-1)$-dimensional $C^{1,\alpha}$ submanifold of $U(0,1)$ containing the origin, we make the following definitions. Let

$$Y = \operatorname{Tan}(N,0),$$

$$K_1(N) = \inf\{t: |Y^{\perp}(y-b)| \leqslant t|y-b| \text{ whenever } y, b \in N\},$$

$$K_2(N) = \sup\{\|\operatorname{Tan}(N,b) - Y\|: b \in \mathbf{N}\},$$

$$K_3(N) = \inf\{t: \|\operatorname{Tan}(N,b) - \operatorname{Tan}(N,y)\| \leqslant t|y-b|^{\alpha} \text{ whenever } y, b \in N\},$$

$$K_4(N) = \sup\{|Y^{\perp}(b)|: b \in N\}, \quad \text{and}$$

$$\eta(N) = \max\{K_i(N): i = 1,2,3,4\}.$$

If N is the graph of a $C^{1,\alpha}$ function $f: Y \cap U(0,1) \to Y^{\perp}$ we set

$$\omega = \{(a,b): \{a + v: v \in Y^{\perp}\} \cap N = \{b\}\},$$

$$\zeta(a) = |a - \omega(a)| \quad \text{whenever } a \in \operatorname{domain} \omega, \quad \text{and}$$

$$\chi(a) = Y + \{t(a - \omega(a)): t \in \mathbf{R}\} \in \mathbf{G}(n,k) \quad \text{whenever } a \notin N.$$

THE VARIFOLD THEOREM. *Suppose* $0 < \alpha < 1$. *There exists* $\beta > 2$ *depending on* n *and* $n-k$ *such that for each* ε *with* $0 < \varepsilon < 1$ *there exists* $\delta > 0$ *and* $0 < C < \infty$ *with the following property*: *If*

(H1) *V is a k-dimensional varifold in* $\mathbf{R}^n$;

(H2) $\|V\|(\mathbf{R}^n \sim U(0,1)) = 0$;

(H3) $\operatorname{spt}\|V\| \cap U(0,\delta) \neq \varnothing$;

(H4) $\|V\|U(0,1) \leqslant (1+\delta)\boldsymbol{\alpha}(k)/2$;

(H5) $\Theta^k(\|V\|, x) \geqslant 1$ *for* $\|V\|$ *almost all* $x \in \mathbf{R}^n$;

(H6) *N is a $(k-1)$-dimensional $C^{1,\alpha}$ submanifold of* $\mathbf{R}^n$ *with the Hölder constant of the derivative not exceeding* $2/\beta$;

(H7) $\|V\|(N) = 0$;

(H8) $0 \in N$;

(H9) $Y = \operatorname{Tan}(N,0)$;

(H10) *N is the graph of a $C^{1,\alpha}$ function $f: Y \to Y^{\perp}$*;

(H11) (*Closure of* N) $\cap\, U(0,1) = N$;

(H12) $\mathscr{H}^{k-1}(N \cap U(0,1)) \leqslant (1+\delta)\boldsymbol{\alpha}(k-1)$;

(H13) $0 \leqslant \eta(N) \leqslant \delta$;

(H14) *V is stationary with respect to vector fields g with* $\operatorname{spt} g \subset U(0,1) \sim N$; *and*

(H15) $V = v(Q)$, *where* $Q \in \mathscr{R}_k(\mathbf{R}^n)$ *is a representative modulo two*, $(\operatorname{spt} \partial Q) \cap U(0,1) = N$ *and* $\|Q\|^2 U(0,1) \leqslant \|P\|^2 U(0,1)$ *whenever* $P \in \mathscr{R}_k(\mathbf{R}^n)$, $\partial P = \partial Q$ *and* $P \llcorner (\mathbf{R}^n \sim U(0,1)) = Q \llcorner (\mathbf{R}^n \sim U(0,1))$;

then

(C1) *for each* $b \in N \cap U(0, 1-\varepsilon)$ *there exists a unit vector* $u_b \in \operatorname{Nor}(N,b)$ *for which*

$$\operatorname{Tan}(\operatorname{spt}\|V\|, b) = \{y + tu_b: y \in \operatorname{Tan}(N,b) \text{ and } t \geqslant 0\};$$

(C2) $\mu = \int |T^\perp(x)|^2 d\|V\|x \leqslant \varepsilon$ *where* $T = \{y + tu_0 \colon y \in \operatorname{Tan}(N, 0)$ *and* $t \in \mathbf{R}\}$;

(C3) T *projects* $\operatorname{spt}\|V\| \cap U(0, 1-\varepsilon)$ *univalently into* T;

(C4) $\|\operatorname{Tan}(\operatorname{spt}\|V\|, x) - \operatorname{Tan}(\operatorname{spt}\|V\|, a)\| \leqslant C \sup\{\mu, \eta(N)\}|x - a|^\alpha$ *whenever* x, $a \in \operatorname{spt}\|V\| \cap U(0, 1-\varepsilon) \sim N$;

(C5) *If* y, $b \in N$ *and* T_b, $T_y \in \mathbf{G}(n, k)$ *containing* $\operatorname{Tan}(\operatorname{spt}\|V\|, b)$ *and* $\operatorname{Tan}(\operatorname{spt}\|V\|, y)$ *respectively, then* $\|T_b - T_y\| \leqslant C \sup\{\mu, \eta(N)\}|y - b|^\alpha$; *and*

(C6) $\|\operatorname{Tan}(\operatorname{spt}\|V\|, x) - T_{\omega(x)}\| \leqslant C \sup\{\mu, \eta(N)\}|x - \omega(x)|^\alpha$ *whenever* $x \in \operatorname{spt}\|V\| \cap U(0, 1-\varepsilon) \sim N$.

PROOF. The proof parallels that of a boundary regularity result of Allard [**5**, §4]. Here we outline the changes and concentrate on places where significant changes are made. A detailed version of our parallel proof is available from the author upon request. The main differences are that Allard's boundary is C^2 rather than $C^{1,\alpha}$, while his variational condition on V is weaker than our (H14) and (H15). We replace Allard's nearest point retraction onto the boundary, ξ, which is no longer Lipschitz, by our ω. We replace Allard's distance ρ from a point to the boundary by our ζ, κ by K_1, $1 - k/p$ by α, and B by N. $R = K_1^{-1}$.

The proof requires several basic properties of η and ω, which we now prove.

(1) $\eta(N) \leqslant \beta C$, where C is the Hölder constant of Df and $\beta > 2$ is a constant depending on n and k.

Indeed, we check that $K_2(N) \leqslant C$ and $K_3(N) \leqslant C$ by first noting that

$$\operatorname{Tan}(N, b) = \{y + \langle y, Df(Y(y))\rangle : y \in Y\},$$

and then applying [**4**, 8.8(5)] to obtain

$$\|\operatorname{Tan}(N, b) - \operatorname{Tan}(N, y)\| \leqslant \|Df(Y(b)) - Df(Y(y))\| \leqslant C|b - y|^\alpha$$

and

$$\|\operatorname{Tan}(N, b) - Y\| \leqslant C|b|^\alpha \leqslant C$$

whenever $b, y \in N$. To see that $K_1(N)$ and $K_2(N)$ do not exceed βC, we let b and y be in N and use the mean value theorem applied to the coordinate functions of f to find $x_1, \ldots, x_{n-k}$ on the line between $Y(y)$ and $Y(b)$ such that

$$f_i(Y(y)) - f_i(Y(b)) = \operatorname{grad} f_i(x_i) \cdot [Y(y) - Y(b)].$$

There is a constant A depending on n and k such that

$$\begin{aligned}
|Y^\perp(y) - Y^\perp(b)| &= |f(Y(y)) - f(Y(b))| \\
&\leqslant \left[\sum_{i=1}^{n-k} |\operatorname{grad} f_i(x_i)|\,|Y(y) - Y(b)|^2\right]^{1/2} \\
&\leqslant \left[\sum_{i=1}^{n-k} |Df(x_i)|^2\right]^{1/2} |y - b| \\
&\leqslant \left[A^2 \sum_{i=1}^{n-k} \|Df(x_i)\|^2\right]^{1/2} |y - b| \leqslant AC(n-k)|y - b|
\end{aligned}$$

since $|x_i| \leqslant 1$. Let $\beta = \max\{A(n-k), 2\}$.

Note that [**4**, 8.8(3)] implies

$$\|\mathrm{Nor}(N, b) - \mathrm{Nor}(N, y)\| = \|\mathrm{Tan}(N, b) - \mathrm{Tan}(N, y)\| \leqslant \eta(N)|b - y|^\alpha$$

whenever $y, b \in N$.

(2) For each r, $0 < r < 1$, let $\mu(r^{-1})$: $\mathbf{R}^n \to \mathbf{R}^n$ be the homothety $\mu(r^{-1})(y) = r^{-1}y$. Let $N_r = \mu(r^{-1})N \cap U(0,1)$. Then $Y = \mathrm{Tan}(N_r, 0)$ and $\eta(N_r)$ decreases to zero as r does.

(3) If $\eta(N) < 1$, then Y maps $N \cap U(0,1)$ diffeomorphically onto $Y(N \cap U(0,1))$ and the Lipschitz constant of the inverse map does not exceed $(1 - [\eta(N)]^2)^{-1/2}$.

Indeed, whenever $y, b \in n \cap U(0,1)$, we have

$$\begin{aligned} |Y(y) - Y(b)| &= |y - b|^2 - |Y^\perp(y - b)|^2 \\ &\geqslant |y - b|^2 - \left[K_1(N)\right]^2 |y - b|^2 \\ &\geqslant \left(1 - [\eta(N)]^2\right)|y - b|^2. \end{aligned}$$

Note that

(4) $|\omega(x) - \omega(a)| \leqslant (1 - [\eta(N)]^2)^{-1/2}|x - a|$ whenever $x, a \in \{z\colon Y(z) \in N \cap U(0,1)\}$.

Allard's proof is a series of lemmas most of which go through in our case with references to Allard's 2.1 and 2.2 replaced by our definitions of η, ω, ζ, and χ and by (1)–(4) above; with 3.4(1) replaced by 5.3.2(1), 3.4(2) by 5.3.2(2), 3.5(1) by 5.3.3(1), 3.5(2) by 5.3.5(1), and 3.5(3) by 5.3.5(2).

In [**5**, 4.4], the inequalities of the proof require minor changes when our references are used, but it is easily checked that results (1)–(4) of [**5**, 4.4] remain as stated. Inequality (5) must be modified to

(5) for small t,

$$\int_{T(M) \cap \{z : |Y(z)| < t\}} |f|^2 \, d\mathscr{H}^k \leqslant \left(2t^2 + 4\eta(N)t\right) \int_{T(M)} \|Df\|^2 \, d\mathscr{H}^2 + 4\alpha(k-1)[\eta(N)]^2 t.$$

This is verified by following the computation of Allard [**5**, pp. 434-435], but using the fact that by definition of $\eta(N)$, $|Y^\perp(z)| \leqslant \eta(N)$ whenever $z \in N$.

The proof of [**5**, Lemma 4.5] requires significant change. We now prove a replacement lemma.

LEMMA. *There exist constants $\delta > 0$ and C such that whenever V and N satisfy* (H1)–(H15) *for δ, $T \in \mathbf{G}(n, k)$ containing $Y = \mathrm{Tan}(N, 0)$, and*

$$\mu = \left(\int |T^\perp(x)|^2 d\|V\|x\right)^{1/2},$$

then

$$\int_{U(0, 15/16) \times \mathbf{G}(n,k)} \|S - T\|^2 \, dV(x, S) \leqslant C \sup\{\mu^2, \eta^2(N)\}.$$

PROOF. Let ϕ: $U(0,1) \to \mathbf{R}$ be a smooth function with $0 \leqslant \phi \leqslant 1$, spt $\phi \subset U(0,1)$, and $U(0,15/16) \subset \{x: \phi(x) = 1\}$. Let $L = \sup\{|\text{grad}\, \phi(x)|: x \in U(0,1)\}$. Let $g(x) = [\phi(x)]^2 T^\perp(x)$. Then

$$Dg(x)\cdot S = 2\phi(x) S\left[T^\perp(x)\right]\cdot \text{grad}\,\phi(x)\cdot[\phi(x)]^2 T^\perp \cdot S.$$

From [4, 8.9(3) and (2)] we have

$$\begin{aligned}[\phi(x)\|S-T\|]^2 = [\phi(x)]^2\|T^\perp\cdot S\|^2 &\leqslant Dg(x)\cdot S - 2\phi(x)S\left[T^\perp(x)\right]\cdot \text{grad}\,\phi(x)\\ &\leqslant Dg(x)\cdot S + 2\phi(x)L\|S-T\|\,|T^\perp(x)|\end{aligned}$$

whenever $(x, S) \in U(0,1)\times \mathbf{G}(n,k)$. As in the proof of Proposition 3 of §5.3, condition (H15) of the Varifold Theorem implies that

$$\left|\int Dg(x)S\,dV(x,S)\right| \leqslant \int |g|\,d\|N\|.$$

This, combined with Schwarz's inequality, implies

$$(*)\qquad \int[\phi(x)\|S-T\|]^2\,dV(x,S) \leqslant \int[\phi(x)]^2|T^\perp(x)|\,d\|N\|x + 2L\left(\int[\phi(x)\|S-T\|]^2\,dV(x,S)\right)^{1/2}\left(\int|T^\perp(x)|^2\,dV(x,S)\right)^{1/2}.$$

Let $z = (\int[\phi(x)\|S-T\|]^2\,dV(x,S))^{1/2}$. Solving the polynomial equation

$$z^2 - 2L\left(\int|T^\perp(x)|^2\,dV(x,S)\right)^{1/2} z - \int[\phi(x)]^2|T^\perp(x)|\,d\|N\|x = 0$$

for z yields two roots $r_1 < r_2$. It is easy to see that $(*)$ is satisfied if and only if $r_1 \leqslant z \leqslant r_2$. Computing r_2 and noting that $(a^2+b^2)^{1/2} \leqslant a + b$ for positive a and b, we have

$$\begin{aligned}&\left(\int[\phi(x)]^2\|S-T\|^2\,dV(x,S)\right)^{1/2}\\ &\quad\leqslant r_2 = 2L\left(\int|T^\perp(x)|^2\,dV(x,S)\right)^{1/2}\\ &\qquad + \left(L^2\int|T^\perp(x)|^2\,dV(x,S) + \int[\phi(x)]^2|T^\perp(x)|\,d\|N\|x\right)^{1/2}\\ &\quad\leqslant 3L\left(\int|T^\perp(x)|^2\,dV(x,S)\right)^{1/2} + \left(\int|T^\perp(x)|\,d\|N\|x\right)^{1/2}.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{U(0,15/16)\times\mathbf{G}(n,k)}\|S-T\|^2\,dV(x,S)\\ &\quad\leqslant \int\phi^2(x)\|S-T\|^2\,dV(x,S)\\ &\quad\leqslant 9L^2\mu^2 + \int|Y^\perp(x)|\,d\|N\|x + 6L\mu\left(\int|Y^\perp(x)|\,d\|N\|x\right)^{1/2}\\ &\quad\leqslant 9L^2\mu^2 + 2\boldsymbol{\alpha}(k-1)\eta^2(N) + 6L\mu\eta(N).\end{aligned}$$

We may choose ϕ so that $L \leqslant 2/16$. If $\delta < 1$, then $\eta^2(N) < \eta(N)$. Let $C = 9L^2 + 2\alpha(k-1) + 6L$. $\square$

The change in (5) above and our references yield minor changes in the constants in the proofs of Allard's [**5**, 4.7 and 4.8]; however, the results go through as stated.

To prove the Varifold Theorem, we fix ε and γ with $0 < \varepsilon < \gamma < 1$ and choose δ sufficiently small that

$$2^{1/2}\varepsilon^{-(k+2)/2}\left(1 + (2\alpha(k))^{1/2}(1+\delta)\right)\delta$$

does not exceed the δ of [**5**, 4.8]. Suppose V, N and Q are as in the theorem. $\operatorname{Tan}(\operatorname{spt}\|V\|, 0)$ is a half-plane of some T in $\mathbf{G}(n, k)$ and $\operatorname{Tan}(N, 0) \subset T$. Whenever $b \in N$, let $T_b = \operatorname{Tan}(N, b) + [\operatorname{Nor}(N, b) \cap T]$. Then

$$\begin{aligned}\|T_b - T\| &\leqslant \|\operatorname{Tan}(N, b) - \operatorname{Tan}(N, 0)\| \\ &\quad + \|[\operatorname{Nor}(N, b) \cap T] - [\operatorname{Nor}(N, 0) \cap T]\| \\ &\leqslant 2K_3(N)|b|^2 < 2\eta(N).\end{aligned}$$

Whenever $x \in U(0, 1)$,

$$|T_b^\perp(x)| < |T^\perp(x)| + \|T_b - T\|\,|x| \leqslant |T^\perp(x)| + 2\eta(N)|x|.$$

Then, whenever $b \in N \cap U(0, 1 - \gamma)$,

$$\begin{aligned}\mu_b &= \left((1 - |b|)^{-k-2}\int_{U(b, 1-|b|)} |T_b^\perp(x)|^2 d\|V\|x\right)^{1/2} \\ &\leqslant \varepsilon^{-(k+2)/2}\left(2\int_{U(b, 1-|b|)} |T^\perp(x)|^2 d\|V\|x\right. \\ &\qquad \left. + 2\int_{U(b, 1-|b|)} 4\eta^2(N)|x|^2 d\|V\|x\right)^{1/2} \\ &\leqslant \sqrt{2}\,\varepsilon^{-(k+2)/2}\left[\mu + 2\eta(N)(\|V\|U(0, 1))\right]^{1/2} \\ &\leqslant \sqrt{2}\,\varepsilon^{-(k+2)/2}\left[1 + (2\alpha(k))^{1/2}(1+\delta)\right]\delta.\end{aligned}$$

The proof now proceeds as on [**5**, p. 443] with $\xi(x)$, $\rho(x)$, η and $1 - k/p$ replaced by $\omega(x)$, $|x - \omega(x)|$, $\eta(N)$ and α, respectively.

COROLLARY. *Let* $0 < \alpha < 1/2$. *Let* S, $B \subset \mathbf{R}^n$ *be solutions to our variational problem. Suppose* $0 \in B \sim \operatorname{spt}^2(F)$, $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1$ *and* $\Theta^k(\mathscr{H}^k \llcorner S, 0) = 1/2$. *Then the following three statements hold.*

(1) *There exists a unit vector* $u \in \operatorname{Nor}(B, 0)$ *for which* $\operatorname{Tan}(S, 0) = \{y + tu\colon y \in \operatorname{Tan}(B, 0)$ *and* $t \geqslant 0\}$.

(2) *Let* $T = \{y + tu\colon y \in \operatorname{Tan}(B, 0)$ *and* $t \in \mathbf{R}\}$. *There exists an* $R > 0$ *such that* $S \cap U(0, R)$ *is the graph of a* $C^{1,\alpha}$ *function* $f\colon T(S \cap U(0, R)) \to T^\perp$ *and* $U(0, R) \cap B$ *is a* $C^{1,\alpha}$ *manifold.*

(3) *There exists an* $R > 0$, *a rotation* θ *of* $\mathbf{R}^{n-1}$ *and a* $C^{1,\alpha}$ *function*

$$u\colon \left(\mathbf{R}^{k-1} \times (\mathbf{R}^+ \cup \{0\})\right) \cap U(0, R) \to \mathbf{R}^{n-k}$$

such that

$$\operatorname{graph} u = \psi(S) \cap \left(U^k(0, R) \times \mathbf{R}^{n-k}\right)$$

and

$$\operatorname{graph}\left(u|\left(\mathbf{R}^{k-1} \times \{0\}\right) \cap U(0, R)\right) = \psi(B) \cap \left(U^k(0, R) \times \mathbf{R}^{n-k}\right),$$

where ψ: $\mathbf{R}^n \to \mathbf{R}^n$ *is defined by applying* θ *to the variables* $x_1, \ldots, x_{k-1}$, $x_{k+1}, \ldots, x_n$ *and leaving* x_k *fixed.*

PROOF. In light of the introduction to this section only (3) requires proof. Let θ be the rotation of Theorem 5.2. Statement (3) follows provided T is not contained in $Z = q^*(\mathbf{R}^{n-1})$. Suppose that S were tangent to Z at the origin. Let $r > 0$ be such that $S \cap ([T \cap U(0, r)] \times T^{\perp})$ is the graph of a $C^{1,\alpha}$ function on $T \cap U(0, r)$. A standard barrier argument ensures that S lies entirely in $\mathbf{R}^{k-1} \times (\mathbf{R}^{+} \cup \{0\}) \times \mathbf{R}^{n-k}$ and precludes the possibility that S is tangent to Z at any point in the interior of S. Let p be a point in T and $T \cap B(p, s)$ be a disc in T such that $T \cap \partial B(p, s)$ intersects B only at the origin where it is tangent to B. Then the support of $\partial[S^* \llcorner (T \cap B(p, s)) \times T^{\perp}]$ is a $(k-1)$-dimensional surface lying wholly to one side of T except at the origin where it is tangent to T. Thus we choose a C^{∞} $(k-1)$-dimensional manifold N in $(T \cap \partial B(p, s)) \times T^{\perp}$ which lies between $T \cap \partial B(p, s)$ and $\operatorname{spt} \partial[S^* \llcorner (T \cap B(p, s)) \times T^{\perp}]$ and which is tangent to T at the origin. Solving a Dirichlet problem, we find a minimal surface M spanning N. Again, a barrier argument shows that $M \sim \{0\} \subset (T \cap B(p, s)) \times T^{\perp}$ lies strictly between $T \cap B(p, s)$ and $\operatorname{spt}(S^* \llcorner (T \cap B(p, s)) \times T^{\perp})$. M is necessarily tangent to T at the origin. By **[13**, Theorem 2], this is impossible.

5.5. $C^{2,\alpha}$ *regularity of S and B.* We restrict to the case where S has codimension one in $\mathbf{R}^n$.

THEOREM. *Let* S, $B \subset \mathbf{R}^{k-1}$ *be solutions to our variational problem. Suppose that* $0 < \alpha < 1/2$, $0 \in B \sim \operatorname{spt}^2(F)$, $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1$ *and* $\Theta^{k}(\mathscr{H}^{k} \llcorner S, 0) = 1/2$. *Then* $S \cap (U^k(0, R) \times \mathbf{R})$ *is a* $C^{2,\alpha}$ *manifold with* $C^{2,\alpha}$ *boundary* $B \cap (U^k(0, R) \times \mathbf{R})$ *for some* $R > 0$.

NOTATION. The ambient space will be considered as $\mathbf{R}^{k+1} = \mathbf{R}^k \times \mathbf{R}$ and the points of $\mathbf{R}^k$ will be denoted by $(x, t) = (x_1, \ldots, x_{k-1}, t)$. Differentiation with respect to x_i will be D_i. The partial with respect to t will be written D_k or D_t as convenient. We define p_1 and p_2 to be projections of $\mathbf{R}^{k+1}$ onto the first k coordinates and the last coordinate, respectively, while p_1^* and p_2^* will be the associated orthogonal injections. R, ψ and U are as in (3) in the corollary of §5.4,

$$\Sigma = U^k(0, R) \cap \{(x, t) \colon t \geqslant 0\} \quad \text{and} \quad U = \Sigma \cap \{(x, t) \colon t = 0\}.$$

Derivation of the differential equation. Given any θ: $\mathbf{R}^k \to \mathbf{R}$ of class one with support in $U^k(0, R)$ we define

$$h(s, z) = z + s[p_2^* \circ \theta \circ p_1](z)$$

whenever $z \in \mathbf{R}^{k+1}$ and $-\varepsilon < s < \varepsilon$, some small positive ε. We let $h_s(t) = h(s, t)$ and define J: $(-\varepsilon, \varepsilon) \to \mathbf{R}$ by

$$J(s) = \mathscr{H}^k\left(h_s(\psi(S) \cap \Sigma \times \mathbf{R})\right) + c\mathscr{H}^{k-1}\left(h_s(\psi(B) \cap \Sigma \times \mathbf{R})\right).$$

That is,

$$J(s)=\int_{\Sigma}\left(1+\sum_{j=1}^{k}|D_j u(x,t)+sD_j\theta(x,t)|^2\right)^{1/2}d\mathscr{L}^k(x,t)$$

$$+c\int_U\left(1+\sum_{j=1}^{k-1}|D_j u(x,0)+sD_j\theta(x,0)|^2\right)^{1/2}d\mathscr{L}^{k-1}x.$$

By the minimizing properties of S and B, J has a minimum at $s=0$. Thus,

$$\text{(1)}\quad 0=\int_{\Sigma}\sum_{i=1}^{k}\left(1+\sum_{j=1}^{k}|D_j u(x,t)|^2\right)^{-1/2}D_i u(x,t)\cdot D_i\theta(x,t)\,d\mathscr{L}^k(x,t)$$

$$+c\int_U\sum_{i=1}^{k-1}\left(1+\sum_{j=1}^{k-1}|D_j u(x,0)|^2\right)^{-1/2}D_i u(x,0)\cdot D_i\theta(x,0)\,d\mathscr{L}^{k-1}x.$$

This equation may be split into an interior equation and a boundary condition as follows. Consider functions θ: $\mathbf{R}^k\to\mathbf{R}$ with support in $U^k(0,R)$ of the form $\theta(x,t)=\rho(t)\phi(x)$, where ρ: $\mathbf{R}\to\mathbf{R}$ and ϕ: $\mathbf{R}^{k-1}\to\mathbf{R}$ are of class one. For such θ, equation (1) becomes

$$\text{(2)}\quad 0=\int_{\Sigma}\sum_{i=1}^{k-1}\left(1+\sum_{j=1}^{k}|D_j u(x,t)|^2\right)^{-1/2}D_i u(x,t)\cdot D_i\phi(x)\rho(t)\,d\mathscr{L}^k(x,t)$$

$$+\int_{\Sigma}\left(1+\sum_{j=1}^{k}|D_j u(x,t)|^2\right)^{-1/2}D_k u(x,t)\cdot\rho'(t)\phi(x)\,d\mathscr{L}^k(x,t)$$

$$+c\int_U\sum_{i=1}^{k-1}\left(1+\sum_{j=1}^{k-1}|D_j u(x,0)|^2\right)^{-1/2}D_i u(x,0)\cdot D_i\phi(x)\rho(0)\,d\mathscr{L}^{k-1}x.$$

The interior equation is obtained by considering functions ρ which vanish at $t=0$. The boundary condition arises by first letting $p_\varepsilon(t)$ be a smooth function such that $p_\varepsilon(0)=1$, $p_\varepsilon(t)=0$ if $t>2\varepsilon$, and $\rho'(t)<1/\varepsilon$ for all t whenever $\varepsilon>0$ and then taking the limit as ε approaches zero. This yields the equation

$$\text{(I)}\quad 0=\int_{\Sigma}\sum_{i=1}^{k}\left(1+\sum_{j=1}^{k}|D_j u(x,t)|^2\right)^{-1/2}D_i u(x,t)\cdot D_i\theta(x,t)\,d\mathscr{L}^k(x,t)$$

whenever θ has support in $\Sigma\sim\{(x,t)\colon t=0\}$, and the boundary condition

$$\text{(BC)}\quad 0=c\int_U\sum_{i=1}^{k-1}\left(1+\sum_{j=1}^{k-1}|D_j u(x,0)|^2\right)^{-1/2}D_i u(x,0)\cdot D_i\phi(x)\,d\mathscr{L}^{k-1}x$$

$$+\int_U\left(1+\sum_{j=1}^{k}|D_j u(x,0)|^2\right)^{-1/2}D_k u(x,0)\cdot\phi(x)\,d\mathscr{L}^{k-1}x$$

whenever ϕ has support in $U^{k-1}(0,R)$.

Second partials in the x variables. We fix i, $1 \leqslant i \leqslant k-1$, and show that $D_i Du$ exists and is Hölder continuous. Choose $\zeta = (\gamma, 0)$, where γ is a unit $(k-1)$-vector parallel to the x_i-axis in $\mathbf{R}^{k-1}$. For each $h > 0$ define the difference quotient

$$u^h(x,t) = h^{-1}\big[u((x,t) + h\zeta) - u(x,t)\big].$$

The proof will proceed as follows. A device of Lichtenstein yields a linear elliptic system of an equation and a complementing boundary condition which u^h satisfies. Applying a Schauder estimate of [2] we obtain a bound on the derivative and Hölder constant of u^h which is independent of h. An equicontinuity argument then implies the existence and Hölder continuity of $D_i Du$ on U.

Equations for u^h. The device of Lichtenstein is applied to the interior equation as follows. Whenever $0 \leqslant s \leqslant 1$, define

$$A(s) = \int_\Sigma \sum_{i=1}^{k} \left(1 + \sum_{j=1}^{k} |D_j u + shD_j u^h|^2\right)^{-1/2} \left(D_i u + shD_i u^h\right) \cdot D_i \theta \, d\mathscr{L}^k.$$

Noting that $A(0)$ is the right-hand side of (I) above and $A(1)$ is the right-hand side of equation (I) with θ replaced by $\tilde{\theta}(x,t) = \theta(x - h\gamma, t)$ we arrive at the equations

(I-h) $\quad 0 = \int_0^1 A'(s)\, ds,$

$$0 = \int_\Sigma \sum_{j=1}^{k} \left(\int_0^1 \left(1 + \sum_{i=1}^{k} |D_j u + shD_j u^h|^2\right)^{-1/2} ds \right) D_i u^h \cdot D_i \theta \, d\mathscr{L}^k$$
$$- \int_\Sigma \sum_{i,j=1}^{k} \left(\int_0^1 \left(1 + \sum_{l=1}^{k} |D_l u + shD_l u^h|^2\right)^{-3/2} \right.$$
$$\left. \times \left(D_i u + shD_i u^h\right)\left(D_j u + shD_j u^h\right) ds \right) D_j u^h \cdot D_i \theta \, d\mathscr{L}^k$$

whenever $\theta\colon \Sigma \to \mathbf{R}$ has support in the interior of Σ. With the integrals with respect to s as coefficients, equation (I-h) is a linear equation of u^h. When one applies this procedure to the boundary condition for u, one obtains the following linear boundary condition for u^h.

(BC-h)

$$0 = c\int_U \sum_{i=1}^{k-1} \left(\int_0^1 \left(1 + \sum_{j=1}^{k-1} |D_j u(x,0) + shD_j u^h(x,0)|^2\right)^{-1/2} ds \right)$$
$$\times D_i u^h(x,0) \cdot D_i \phi(x) \, d\mathscr{L}^{k-1}x$$
$$- c\int_U \sum_{i,j=1}^{k-1} \left(\int_0^1 \left(1 + \sum_{l=1}^{k-1} |D_l u(x,0) + shD_l u^h(x,0)|^2\right)^{-3/2} \right.$$
$$\left. \times \left(D_i u(x,0) + shD_i u^h(x,0)\right)\left(D_j u(x,0) + shD_j u^h(x,0)\right) ds \right)$$
$$\times D_i u^h(x,0) \cdot D_i \phi(x) \, d\mathscr{L}^{k-1}x$$

(continues)

$$+\int_U\left(\int_0^1\left(1+\sum_{j=1}^k|D_ju(x,0)+shD_ju^h(x,0)|^2\right)^{-1/2}ds\right)$$
$$\times D_ku^h(x,0)\cdot\phi(x)\,d\mathscr{L}^{k-1}x$$
$$-\int_U\sum_{i=1}^k\left(\int_0^1\left(1+\sum_{j=1}^k|D_ju(x,0)+shD_ju^h(x,0)|^2\right)^{-3/2}\right.$$
$$\left.\times\left(D_ku(x,0)+shD_ku^h(x,0)\right)\left(D_iu(x,0)+shD_iu^h(x,0)\right)ds\right)$$
$$\times D_iu^h(x,0)\cdot\phi(x)\,d\mathscr{L}^{k-1}x.$$

The Schauder estimate. We apply the Schauder estimate of [**2**, Theorem 9.1] to equations (I-h) and (BC-h). The interior equation is uniformly elliptic. This follows from the ellipticity of the area integrand (see [**8**, 5.2.17, 5.2.23 and 5.2.15]) and from the fact that we may bound the coefficients (see below). In fact, we may bound the coefficients independent of h, provided $h<1$.

Let $L'(P,\partial)$ and $B'(P,\partial)$ denote differential operators associated with the terms of the interior and boundary equations of highest order. The supplementary condition follows by direct computation when $k=2$ and from the ellipticity when $k>2$.

Next (BC-h) satisfies the complementing boundary condition. Indeed fix $P\in U$. Let e^t denote the interior unit normal to Σ. Fix $\xi\neq 0$, tangent to $\partial\Sigma$ at P. The polynomial $L'(P,\xi+\tau e_t)$ is a quadratic polynomial having precisely one root with positive imaginary part. Call this root τ_0. Thus $M^+(P,\xi,\tau)=\tau-\tau_0$ is a linear polynomial. Then $B'(P,\xi+\tau e_t)$ is a constant, call it Δ, since B has no derivatives of order two in t. Furthermore Δ is nonzero since to highest order, the boundary equation agrees with the minimal surface equation which is elliptic. When $B'(P,\xi+\tau e_t)$ is divided by $M^+(P,\xi,\tau)$ the remainder is this same nonzero constant. Furthermore, by the bounds on the coefficients of (BC-h) found in the next paragraph, Δ is independent of h provided $h<1$.

Now we verify that the coefficients of (I-h) and (BC-h) are bounded independent of h at least if we restrict to h's less than one. Since

$$\left(1+\sum_{j=1}^k\left|D_ju(x,t)+shD_ju^h(x,t)\right|^2\right)^{-1/2}$$

does not exceed one for any s or h, it will suffice to fix j, $1\leqslant j\leqslant k$, and bound $|D_ju(x,t)+shD_ju^h(x,t)|$. We recall that u is $C^{1,\alpha}$ and let K denote the Hölder constant of D_ju. This combined with the definition of u^h implies

$$|D_ju(x,t)+shD_ju^h(x,t)|\leqslant K|(x,t)|^\alpha+sK|(x,t)+h\zeta|^\alpha$$
$$+sK|(x,t)|^\alpha+(1+2s)|D_ju(0,0)|.$$

The bound on $|D_ju(x,t)+shD_ju^h(x,t)|$ now follows since $|(x,t)|<R$, $|\zeta|=1$ and $h<1$. A similar argument bounds the coefficients of (BC-h).

Finally, we show the existence of a constant which bounds the Hölder constants of the coefficients of (I-h) and (BC-h). Note that since each $D_j u$ is $C^{0,\alpha}$ so is each coefficient. Again we consider only those h's which are less than one so that the Hölder constants of the coefficients may be bounded independent of h.

Let $P, Q \in \Sigma$, and compute

$$\left|\left(1+\sum_{j=1}^{k}|D_j u(P)+shD_j u^h(P)|^2\right)^{-1/2}-\left(1+\sum_{j=1}^{k}|D_j u(Q)+shD_j u^h(Q)|^2\right)^{-1/2}\right|$$
$$\leqslant\left|\left(1+\sum_{j=1}^{k}|D_j u(Q)+shD_j u^h(Q)|^2\right)^{1/2}-\left(1+\sum_{j=1}^{k}|D_j u(P)+shD_j u^h(P)|^2\right)^{1/2}\right|$$
$$\leqslant\left|\left(1+\sum_{j=1}^{k}|D_j u(Q)+shD_j u^h(Q)|^2\right)-\left(1+\sum_{j=1}^{k}|D_j u(Q)+shD_j u(P)|^2\right)\right|.$$

Let E be the supremum over $j = 1, 2,\dots,k$ of the bounds on the $|D_j((x, t)) + shD_j u^h((x, t))|$ computed above. Since $a^2 - b^2 \leqslant 2\sup\{a, b\}(a - b)$ and $|a| - |b| \leqslant |a - b|$, we have, for each j,

$$\begin{aligned}|D_j u(Q)+shD_j u^h(Q)|^2-|D_j u(P)+shD_j u^h(P)|^2&\\ \leqslant 2E\big(|D_j u(Q)-D_j(P)|+sh|D_j u^h(Q)-D_j u^h(P)|\big)&\\ \leqslant 2E\big(|D_j u(Q)-D_j(P)|+s|D_j u(Q+h\zeta)-D_j u(P+h\zeta)|\big)&\\ \leqslant 2E(1+s)K|Q-P|^\alpha.&\end{aligned}$$

Thus,

$$\left|\left(1+\sum_{j=1}^{k}|D_j u(P)+shD_j u^h(P)|^2\right)^{-1/2}-\left(1+\sum_{j=1}^{k}|D_j u(Q)+shD_j u^h(Q)|^2\right)^{-1/2}\right|$$
$$\leqslant 2kEK(1+s)|Q-P|^\alpha.$$

Taking the integral with respect to s gives a Hölder constant for the coefficient which depends only on k, E and K. A similar computation applies to the other coefficients.

Thus by [**2**, Theorem 9.1] there is a constant C_1 independent of h such that

$$\sup\left\{d_p|Du^h(P)|: P\in\Sigma\right\}$$
$$+\sup\left\{d_P^{1+\alpha}|Du^h(P)-Du^h(Q)|/|P-Q|^\alpha: 4|P-Q|\leqslant d_P d_Q\right\}$$

does not exceed C_1. Let $R_1 = 4^{-1}R$. Also there is a constant C_2 independent of h such that

$$|Du^h(P)|+\big(|Du^h(P)-Du^h(Q)|/|P-Q|^\alpha\big)\leqslant C_2$$

for all P and Q in $\Sigma \cap U^k(0, R_1)$.

Convergence of sequences of u^h*'s*. Suppose we have a sequence of $g_j = u^{h_j}$ on $\Sigma \cap U^k(0, R_1)$ where $0 < h_j < 1$ for all j and h_j approaches zero as j approaches infinity. By the computations above, both Dg_j and the Hölder constant of Dg_j are bounded independent of j. Furthermore, if T is the real-valued function defined by

$$T(\phi) = \lim_{j \to \infty} \int g_j \phi \, d\mathscr{L}^k$$

whenever ϕ: $\Sigma \cap U^k(0, R_1) \to \mathbf{R}$ is C^∞ with compact support, where the integral is over $\Sigma \cap U^k(0, R_1)$, then T is continuous and linear. Thus by [**8**, 5.2.2], the sequence g_j converges uniformly to a function g which is continuous and has a Hölder constant of exponent α which does not exceed C_2. This g is $D_{x_i} Du$.

Second partials in the t variable. We will use equation (2) to see that $D_t u(x, t)$ equals a function which is $C^{1,\alpha}$ in t for $|t|$ sufficiently small. Since all second partials of u which involve at most one derivative in the t variable exist, we may integrate by parts in the first and third terms, use Fubini's Theorem [**8**, 2.6.2], and then integrate by parts with respect to t in the inner integral of the first term to rewrite equation (2) as

$$\begin{aligned} 0 = -\int_U \Bigg[&\int_{t \geq 0} \int_0^t \sum_{i=1}^{k-1} D_i \left(D_i u(x, s) \left(1 + \sum_{j=1}^{k} |D_j u(x, s)|^2 \right)^{-1/2} \right) ds \rho'(t) \, dt \\ &+ \int_{t \geq 0} D_k u(x, t) \left(1 + \sum_{j=1}^{k} |D_j u(x, t)|^2 \right)^{-1/2} \rho'(t) \, dt \Bigg] \phi(x) \, d\mathscr{L}^{k-1} x \\ &+ c \int_U \sum_{i=1}^{k-1} D_i \left(D_i u(x,0) \left(1 + \sum_{j=1}^{k-1} |D_j u(x,0)|^2 \right)^{-1/2} \right) \rho(0) \phi(x) \, d\mathscr{L}^{k-1} x. \end{aligned}$$

This equation is valid for all smooth ϕ and ρ with compact supports in U and $\mathbf{R}^+ \cup \{0\}$. In particular, if $\rho(0) = 0$, the third term drops out. Thus for $\mathscr{L}^{k-1}$ almost all x in U, the expression in the square brackets is zero. Since all the partials of u appearing in the expression are continuous in x, this expression is zero for all x in U. By the constancy theorem (see [**1**, 3.27]), for each x there is a constant C_x such that

$$\begin{aligned} &D_k u(x, t) \left(1 + \sum_{j=1}^{k} |D_j u(x, t)|^2 \right)^{-1/2} \\ &\qquad = -\int_0^t \sum_{i=1}^{k-1} D_i \left(D_i u(x, s) \left(1 + \sum_{j=1}^{k} |D_j u(x, s)|^2 \right)^{-1/2} \right) ds + C_x \end{aligned}$$

for almost all $t > 0$. Again, by continuity, the equation is valid for all $t \geq 0$.

Since the integrand is C^α in S, the right-hand side of this equation is $C^{1,\alpha}$ in t for each x. Furthermore, all terms appearing on the left besides $D_k u(x, t)$ are known to be $C^{1,\alpha}$ in t. The right-hand side does not equal one for any x and t. Solving for $D_k u(x, t)$ we see that $D_k u(x, t)$ is equal to a function which is $C^{1,\alpha}$ in t for $|t|$ sufficiently small.

Thus u is $C^{2,\alpha}$ in $\Sigma \cap U^k(0, R_2)$ for some $R_2 > 0$.

5.6. *Analyticity of S and B.* Continuing to restrict to the case where S has codimension one, we now prove our final regularity result.

THEOREM. *Let S, $B \subset \mathbf{R}^n$ be solutions to our variational problem. Suppose* $0 \in B \sim \operatorname{spt}^2(F)$, $\Theta^{k-1}(\mathscr{H}^{k-1} \llcorner B, 0) = 1$ *and* $\Theta^k(\mathscr{H}^k \llcorner S, 0) = 1/2$. *Then* $S \cap (U(0, R) \times \mathbf{R})$ *is a real analytic manifold with real analytic boundary* $B \cap (U^k(0, R) \times \mathbf{R})$ *for some* $R > 0$.

PROOF. The function u of the last section is seen to be analytic at the origin by applying a theorem of Morrey [**12**, 6.8.2] to equations (I) and (BC). (I) is verified to be elliptic and the system is verified to satisfy the complementary boundary condition with weights 0, 2 and 0 by a computation nearly identical to those of the previous section.

6. Size of the possible singular set. Returning to arbitrary codimension we now show that the set of points on B which do not satisfy our density hypotheses has Hausdorff dimension at most $k - 2$. We prove a theorem for k-dimensional rectifiable currents in $\mathbf{R}^n$ that when applied to S^* yields conclusion (3) of our Main Theorem.

Whenever $Q \in \mathscr{R}_k(\mathbf{R}^n)$, we define

$$\omega(Q) = \left[\{x: \Theta^k(\|Q\|, x) > 1/2\} \cup \{x: \Theta^{k-1}(\|\partial Q\|, x) > 1\}\right] \cap \operatorname{spt}(\partial Q).$$

THEOREM. *If* $1 \leqslant k \leqslant n$, $T \in \mathscr{R}_k(\mathbf{R}^n)$ *and T is area minimizing modulo two, then* $\mathscr{H}^m(\omega(T)) = 0$ *whenever* $m > \sup\{0, k - 2\}$.

PROOF. We use induction with respect to k. The theorem is known to be true for $k = 1$ [**7**, p. 771].

Assume the theorem is true with k replaced by $k - 1$. Suppose, however, that for T and k as in the statement of the theorem $\mathscr{H}^m(\omega(T)) > 0$ for some $m > \sup\{0, k - 2\}$.

Step 1. Whenever $0 \leqslant m$ and $A \subset \mathbf{R}^n$, we define

$$\phi_\infty^m(A) = \inf\left\{\sum_{B \in G} \boldsymbol{\alpha}(m) 2^{-m} (\operatorname{diam} B)^m : G \text{ is a countable open covering of } A\right\}.$$

The definition of Hausdorff measure implies that

$$\phi_\infty^m(A) = 0 \quad \text{if and only if} \quad \mathscr{H}^m(A) = 0,$$

and [**8**, 2.10.19(2)] implies that $\Theta^{*m}(\phi_\infty^m \llcorner A) \geqslant 2^{-m}$ for $\mathscr{H}^m$ almost all x in A.

Step 2. We choose $a \in \omega(T)$ such that $\Theta^{*m}(\phi_\infty^m \llcorner \omega(T), a) > 0$. Let $d > 0$ be such that $\Theta^{*m}(\phi_\infty^m \llcorner \omega(T), a) > 2^m d > 0$.

Step 3. Construct a sequence similar to [**8**, pp. 624–625] of β_i approaching infinity such that

$$\phi_\infty^m\left[\omega(T) \cap B\left(a, \beta_i^{-1}\right)\right] > 2^m d \beta_i^m \boldsymbol{\alpha}(M)$$

and such that a subsequence of $Q_i = (\mu(\beta_i) \circ \tau(-a))_\# T$ converges to Q, an oriented tangent cone. It is easy to verify that Q is area minimizing modulo two. As in [**7**, Lemma 1]

$$\phi_\infty^m[\omega(Q) \cap B(0,1)] \geqslant \limsup_{i \to \infty} \phi_\infty^m[\omega(Q) \cap B(0,1)] \geqslant 2^m d \boldsymbol{\alpha}(m).$$

Thus $\mathscr{H}^m(\omega(Q)) > 0$.

Step 4. From here proceed as in [**7**, Theorem 1] using our induction assumption.

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